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Kronecker product of Sp(2*n*) representations using generalised Young tableaux

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Abstract. Using generalised Young tableaux, we obtain an explicit formula for the reduction of the Kronecker product of irreducible representations of the symplectic groups. This extends a previous work devoted to the case of orthogonal groups.

1. Introduction

In the present paper, generalising a method introduced by the authors (Girardi *et al* 1982a, b) for the reduction of the Kronecker product of irreducible representations (IRs) of SO(n) groups, we give general formulae for the direct product of two IRs of symplectic groups Sp($2n$). This problem has been studied in the past by Littlewood (1958) and King (1971) using character theory and Schur functions (Littlewood 1950). Recently, Black *et al* (1983) have written an extensive review on the application of S-function techniques to the evaluation of Kronecker products of IRs of compact semisimple Lie groups. Another approach was taken by Fischler (1980) who used Young tableaux methods but his method, despite its numerous rules, gives in the general case ambiguous results. The method we propose makes use of the weight vectors and reduces the problem to evaluating a sum of products of generalised Young tableaux (GYTs). GYTs are tableaux which can include negative boxes; their definitions and product rules were defined in Girardi *et al* (1982a, b). The use of these GYTs provides a consistent tool for the reduction of Kronecker products of IRs of classical groups, and whether it is relevant for exceptional groups is now under study. Note that the expression 'generalised Young tableau' had already been introduced by King (1970) in a different sense for the study of mixed representations of the full linear group. In § 2 we give a very short recall of Sp($2n$) groups; in § 3 we present our result and finally in § 4 we give an illustrating example.

2. A reminder of Sp(2*n*) groups

The Lie algebra of Sp($2n$) groups can be realised with the help of $n(2n+1)$ infinitesimal generators Z^i_j (Gilmore 1970)

$$[Z^i_j, Z^r_s] = \mathrm{sgn}(jr)\{Z^i_s\delta^{-j}_{-r} + Z^{-j}_{-r}\delta^i_s + Z^i_{-r}\delta^{-j}_s + Z^{-j}_s\delta^r_{-i}\}, \qquad i, j, r, s = \pm 1, \pm 2, \ldots, \pm n,$$

$$Z^i_j = Z^{j\dagger}_i = -\mathrm{sgn}(ij)Z^{-j}_{-i}.$$

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If we relabel the generators $Z_{j'}^{i'}$ as

$$Z_{j'}^{i'} \to Z_{j}^{i}, \qquad i = \begin{cases} 2i', & i' > 0, \\ 2(-i') - 1, & i' < 0, \end{cases}$$

the n generators $H_i = Z_{2i}^{2i}$ $(i = 1, \ldots, n)$ commute with each other and span the maximal abelian (Cartan) subalgebra of Sp$(2n)$. The IRs of Sp$(2n)$ can be labelled by n positive integers μ_i $(\mu_i \geqslant \mu_{i+1})$ which are the highest eigenvalues (weight) of the generators H_i. Hereafter an IR will be labelled by the n-tuple $(\mu_1, \mu_2, \ldots, \mu_n)$ or $[\mu]$. The dimensionality of the IR $[\mu]$ is

$$N[\mu] = P(l)/P(\tau) \qquad \text{where } l_i = \mu_i + \tau_i,\ \tau_i = n - i + 1,$$

and P is a product of the form

$$P(l) = \prod_{j=1}^{n} l_j \prod_{i \geqslant k}^{n} (l_k - l_i)(l_k + l_i).$$

Another way of labelling Sp$(2n)$ IRs is by n non-negative integers q_i (Dynkin notation) which are related to ours as follows:

$$q_i = \mu_i - \mu_{i+1} \qquad (i = 1, \ldots, n-1), \qquad q_n = \mu_n.$$

Tables of dimensions of IRs of Sp$(2n)$ can be found in Wybourne (1970 and references therein) and Patera and Sankoff (1972).

3. Rules for the product of two Sp(2n) IRs

The general formula for the product of two IRs $[\mu]$ and $[\nu]$ can be written as

$$[\mu] \times [\nu] = \Sigma_1 + \Sigma_2, \tag{3.1}$$

$$\Sigma_1 = \sum_{k=0}^{N} (P_n^{2k} \times [\nu])_{\mathrm{A}} \times [\mu] - \sum_{k=a}^{N} (P_n^{2a} \times [\mu])_{\mathrm{NA}} \times (P_n^{2(k-a)} \times [\nu])_{\mathrm{A}}, \tag{3.2}$$

$$\Sigma_2 = \sum_{k=2}^{N} \sum_{r=0}^{k-2} (-1)^{k-r} (P_n^{2r} \times [\nu])_{\mathrm{A}} \times [[\mu] + \tilde{P}_n^{2(k-r)}], \tag{3.3}$$

where P_n^{2k} is a sum of n-row negative GYT, $P_n^{2k} = \Sigma_{\{\alpha\}} P_n^{2k}(\alpha)$, $P_n^{2k}(\alpha) = [0, \ldots, -\alpha_2, -\alpha_1]$ in which the α_i are positive even numbers such that $\alpha_i \geqslant \alpha_{i+1}$ and $\Sigma_{i=1}^{n} \alpha_i = 2k$. When P_n^{2k} acts on the IR $[\nu]$ the $\{\alpha_i\}$ must satisfy $\alpha_\nu \leqslant 2\nu_i$. For a given k there are several possible sets $\{\alpha_i\}$, except for the extreme case

$$k_{\max} = N \equiv \sum_{i=1}^{n} \nu_i \tag{3.4}$$

where

$$P_n^{2N} = [-2\nu_n, \ldots, -2\nu_2, -2\nu_1] \tag{3.5}$$

is uniquely defined.

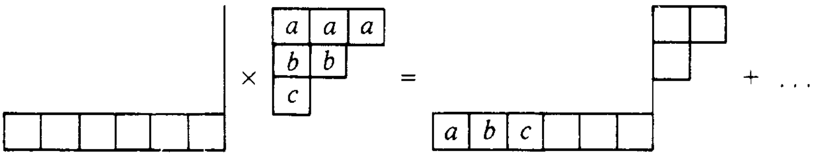
The label a is the smallest integer such that $P_n^{2a} \times [\mu]$ gives a 'not-allowed' (NA) GYT (see definition below). Actually

$$a = \mu_n + 1. \tag{3.6}$$

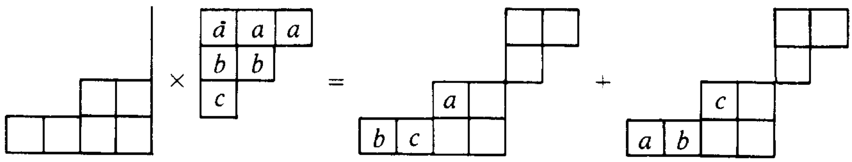
The subscript 'A' (allowed) means that among the GYTs $[\lambda]=[\lambda_1, \lambda_2, \ldots, \lambda_n]$ appearing in the product, one has to keep all those which fulfil the conditions:

(i) $$\lambda_{n-i+1} \geqslant -\tfrac{1}{2}\alpha_i; \tag{3.7}$$

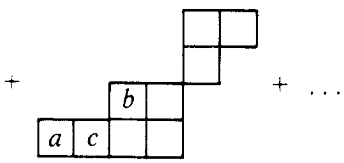
(ii) a GYT $[\lambda]$ coming from $P_n^{2k}(\alpha)\times[\nu]$ is not an allowed one if there exists a GYT $[\lambda']$, coming from $P_n^{2k}(\alpha')\times[\nu]$, with the same positive part (i.e. $\lambda_1=\lambda'_1, \ldots, \lambda_i=\lambda'_i$ if $\lambda_i, \lambda'_i>0$ and $\lambda_{i+1}, \lambda'_{i+1}\leqslant 0$) and with identical 'lexical order' for the negative part, and such that the negative part of $[\lambda']$ is larger than the negative part of $[\lambda]$ (i.e. $|\lambda'_n|>|\lambda_n|$, or $\lambda'_n=\lambda_n$ and $|\lambda'_{n-1}|>|\lambda_{n-1}|$, and so on) or equal to it (i.e. $\lambda'_n=\lambda_n, \ldots, \lambda'_{i+1}=\lambda_{i+1}$). Calling $a, b, \ldots$ the boxes in the 1st, 2nd, ... rows of $[\nu]$, the tableaux $[\lambda]$ and $[\lambda']$ will be said to have the same 'lexical order' if in the negative part of both tableaux we find the letters $a, b, \ldots$ appearing in identical order when reading from left to right and from top to bottom; let us illustrate this condition on an example in Sp(8) where $[\nu]=[3210]$. With P_4^6 $(000-6)$ and P_4^6 $(00-2-4)$ we obtain



$[000-6]$ $[3210]$ $=$ $[210-3]$ $+\ \ldots$ (3.8)



$[000-2-4]$ $\times$ $[3210]$ $=$ $[21-1-2]$ $+$ $[21-1-2]$



$+$ $[21-1-2]$ $+\ \ldots.$ (3.9)

Following the above rule, the first GYT $[\lambda]=[21-1-2]$ appearing in the decomposition of the second product is *not* an allowed GYT, since the GYT $[\lambda']=[210-3]$ appearing in the first product has the same positive part as $[\lambda]$, a negative part larger than that of $[\lambda]$ ($|\lambda'_4|>|\lambda_4|$) and the same lexical order as $[\lambda]$: 'a, b, c'.

Actually, condition (i) guarantees that the set of states associated to the GYT $[\lambda]$ coming from the product $P_n^{2k}(\alpha)\times[\nu]$ really belongs to $[\nu]$, while condition (ii) arises because the states associated to the GYT $[\lambda']$ are in fact already contained in the GYT $[\lambda]$.

The subscript 'NA' means that in the product, one has to keep only the GYTs $[\lambda]$ which do not satisfy conditions (i) and (ii).

In (3.3) $[[\mu]+\tilde{P}_n^{2(k-r)}]$ is a sum of GYTs whose rows are the rows of $[\mu]$ plus the rows of a negative GYT $\tilde{P}_n^{2(k-r)}(\{\gamma_i\})$ which can be computed by the following formula:

$$\tilde{P}_n^{2l}=\sum_{\{\gamma_i\}}\tilde{P}_n^{2l}(\{\gamma_i\})=\sum_{j=1}^{l}\sum_{\{\alpha_i\}}P_n^{2j}(\{\alpha_i\})\times\tilde{P}_n^{2(l-j)}\times(-1)^{j-1} \tag{3.10}$$

where $l=1,2,\ldots$ and $\tilde{P}_n^0=\mathbb{1}=[0,0,\ldots,0]$. (3.11)

The terms given by (3.10) are present only when the IR $[\mu]$ has, at least, as many null labels as the number of negative rows of $\tilde{P}_n^{2(k-r)}(\{\gamma_i\})$. We give explicitly the $\tilde{P}_n^{2l}(\{\gamma_i\})$ for $l=1,2,3$:

$$\begin{aligned}&\tilde{P}_n^2=[0,\ldots,0,-2]=P_n^2, \qquad \tilde{P}_n^4=[0,\ldots,0,-1,-3],\\&\tilde{P}_n^6=[0,\ldots 0,-3,-3]+[0,\ldots,0,-1,-1,-4].\end{aligned} \tag{3.12}$$

On the RHS of (3.2) and (3.3), one has to keep all the GYTs $[\lambda]$ such that $\lambda_i\geqslant 0$ for $i=1,\ldots,n$.

For practical purposes it is more convenient to choose for $[\nu]$ the IR which has the least number of boxes, this minimising the number of operators P^{2k} to be considered.

We note that if $a=1$ $(\mu_n=0)$, equations (3.1)–(3.3) can be rewritten in the more compact form

$$[\mu]\times[\nu]=\sum_{k=0}^{N}\sum_{r=0}^{k}(-1)^{k-r}(P_n^{2r}\times[\nu])_{\mathrm{A}}\times[[\mu]+\tilde{P}_n^{2(k-r)}]. \tag{3.13}$$

Note also that in the case where $[\nu]$ is such that $P_n^{2k}\times[\nu]$ never gives a not-allowed GYT (then $[\nu]=[\nu_1,\nu_1,\ldots,\nu_1]$), equations (3.1)–(3.3) can be rewritten in the following simpler form:

$$[\mu]\times[\nu]=\sum_{k=0}^{N}\{(P_n^{2k}\times[\nu])_{\mathrm{A}}\times[\mu]-(P_n^{2k}\times[\mu])_{\mathrm{NA}}\times[\nu]\}. \tag{3.14}$$

Finally we give compact formulae which can be deduced for completely symmetric or antisymmetric IRs $(\mu\geqslant\nu)$:

$$[\mu,0,\ldots,0]\times[\nu,\ldots,0]=\sum_{l=0}^{\nu}\sum_{k=0}^{\nu-l}[\mu+\nu-2k-2l,k,0,\ldots,0], \tag{3.15}$$

$$[\alpha,\ldots,\alpha]\times[\beta,\ldots,\beta]=\sum_{0\leqslant k_1\leqslant k_2\leqslant\ldots\leqslant k_n}[\alpha+\beta-2k_1,\alpha+\beta-2k_2,\ldots,\alpha+\beta-2k_n]. \tag{3.16}$$

4. An illustrative example

Let us calculate in Sp(10) the Kronecker product $[22000]\otimes[21000]$ of respective dimensions 780 and 320. Hereafter the zero labels will be ignored when unnecessary. We shall operate on $[\nu]=[21]$; since it has only three boxes, we have to consider $k=0,1,2,3$.

$k=0$: $[22]\times[21]=[43]+[421]+[331]+[322]+[3211]+[2221]$. (4.1)

$k=1$: there $P_5^2=[0000-2]$ so

$$(P_5^2\times[\nu])_{\mathrm{A}}=([0000-2]\times[21])_{\mathrm{A}}=[2000-1]+[1100-1]+[10000]. \tag{4.2}$$

Then the products with $[\mu]=[22]$ give

$$\begin{aligned}&[2000-1]\times[22]=[41]+[32]+[311]+[221],\\&[1100-1]\times[22]=[32]+[311]+[221]+[2111],\\&[1]\times[22]=[32]+[221].\end{aligned} \tag{4.3}$$

Now we consider if there are not-allowed terms of the type $P_5^2\times[\mu]$, and find one:

$$(P_5^2\times[\mu])_{\mathrm{NA}}\times[\nu]=[2200-2]_{\mathrm{NA}}\times[21]=[32]+[221]. \tag{4.4}$$

So from the terms obtained in (4.3) one subtracts the contribution of (4.4).

$k=2$: We have two different P_5^4, namely $[000-2-2]$ and $[0000-4]$:

$$([000-2-2]\times[21])_{\mathrm{A}}=[100-1-1]+[1000-2]+[0000-1], \tag{4.5}$$

$$([0000-4]\times[21])_{\mathrm{A}}=[1000-2]. \tag{4.6}$$

In view of rule (ii) and equation (3.7) we ignore (4.6) and only consider (4.5) for multiplying by $[\mu]$:

$$\begin{aligned}&[100-1-1]\times[22]=[21]+[111], \qquad [1000-2]\times[22]=[3]+[21],\\&[0000-1]\times[22]=[21].\end{aligned} \tag{4.7}$$

There is one not-allowed contribution of the type occurring in (3.9); this gives

$$[2200-2]_{\mathrm{NA}}\times[2000-1]=[21] \tag{4.8}$$

to be subtracted from (4.7).

$k=3$: At this level we have only to consider $P_5^6=(000-2-4)$ (from equation (3.5)); this gives

$$\{[000-2-4]\times[21]\}\times\{[22]\}=[000-1-2]\times[22]=[1]. \tag{4.9}$$

Gathering all the terms obtained we have as our final result:

$$\begin{aligned}[22]\times[21]= {} & [43] + [421] + [331] + [322] +[3211]+[2221]\\ 780\times 320 = {} & 42240+71500+35640+28028+32340+ 9152\\ & + [41] + 2[32] + 2[311] + 2[221] +[2111]\\ & \quad 4928+2\times4620+2\times4212+2\times2860+ 1408\\ & + [3] + 2[21] +[111]+[1]\\ & \quad 220+2\times320+ 110 +10.\end{aligned}$$

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Appendix. Product of two GYTs

The product of two GYTs has been defined in Girardi *et al* (1982a) as a generalisation of the product of twoYoung tableaux for U(n) groups. In particular, in the case where the two GYTs $[\lambda]$ and $[\mu]$ to be multiplied have only positive rows, one has to use only the rules for U(n) Young tableaux, the only difference being that one *cannot* replace here a tableau $[\lambda_1, \ldots, \lambda_n]$ with $\lambda_1 \geqslant \ldots \geqslant \lambda_n > 0$ by the tableau $[\lambda_1 - \lambda_n, \ldots, \lambda_{n-1} - \lambda_n, 0]$.

Hereafter, we would like to propose a different way to make the product of two GYTs, equivalent of course to the method already given, but which may be easier to remember once one knows the usual method for U(n) groups.

Let us suppose $[\lambda]$ and $[\mu]$ such that $\lambda_n \leqslant 0$ and $\mu_n \leqslant 0$. Then a direct way to make the product $[\lambda] \times [\mu]$ is to perform the product $[\lambda'] \times [\mu']$ of the two Young tableaux $[\lambda']$ and $[\mu']$ with $\lambda'_i = \lambda_i - \lambda_n$ and $\mu'_i = \mu_i - \mu_n$ $(i = 1, \ldots, n)$, and then to replace each obtained Young tableau $[\nu']$ by the GYT $[\nu]$ such that $\nu_i = \nu'_i + \lambda_n + \mu_n$.

However, we remark that this method may lead to cumbersome calculations. In particular, the product of a totally positive GYT by a totally negative one is more rapidly performed using the first rule already proposed.

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